Fourier Analysis

Feb 08, 2024

Review.

Let f be an integrable function on the circle. Then

① $\|f - S_N f\| \to 0$ as $N \to \infty$

(Recall $\|f\| := \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx}$

and $S_N f(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$.)

② Parseval identity

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Example 1. Let $f(x) = |x|$ on $[-\pi, \pi]$.

By a direct calculation,

$$f(x) \sim \frac{\pi}{2} + \sum_{n=-\infty}^{\infty} \frac{-2}{\pi(2n-1)^2} e^{i(2n-1)x}, \quad x \in [-\pi, \pi]$$

Notice that $\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|^2 dx = \pi^2/3$.

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{\pi^2}{4} + \sum_{n=-\infty}^{\infty} \frac{4}{\pi^2 (2n-1)^4}$$

$$= \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} .$$

By Parseval identity, we have

$$\frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^2}{3} .$$

Then $\displaystyle \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \left(\pi^2/3 - \pi^2/4 \right) \cdot \frac{\pi^2}{8}$

$$= \frac{\pi^4}{96} .$$

From this, we can derive a formula for

$$\sum_{n=1}^{\infty} \frac{1}{n^4} .$$

Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{(2n)^4} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

$$= \frac{1}{2^4} \cdot \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{\pi^4}{96}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96} \Big/ \left(1 - \frac{1}{2^4}\right) = \frac{\pi^4}{90} .$$

(Riemann–Lebesgue Lemma)

Corollary 2. Let f be integrable on the circle.

Then
$$\hat{f}(n) \to 0 \quad \text{as} \quad |n| \to +\infty.$$

Moreover
$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$
converge to 0 as $|n| \to +\infty$.

Pf. Since $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|^2 < \infty$,

it follows that
$$\hat{f}(n) \to 0 \quad \text{as} \quad |n| \to +\infty.$$

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \int_{-\pi}^{\pi} f(x) \frac{e^{inx} + e^{-inx}}{2} dx$$
$$= \pi \left(\hat{f}(n) + \hat{f}(-n) \right) \to 0$$
as $|n| \to \infty$.

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \int_{-\pi}^{\pi} f(x) \frac{e^{inx} - e^{-inx}}{2i} dx$$
$$= \frac{\pi}{i} \hat{f}(n) - \frac{\pi}{i} \hat{f}(-n) \to 0. \quad \blacksquare$$

2. A local convergence Thm.

Thm 3. Let f be integrable on the circle.

Suppose f is differentiable at x_0.

Then $S_N f(x_0) \longrightarrow f(x_0)$ as $N \to \infty$.

(Recall a previous result: Let f be cts on the circle. Suppose $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then

$$S_N f(x) \rightrightarrows f(x) \quad \text{on the circle.})$$

Pf of Thm 3.

Recall that

$$S_N f(x_0) = f * D_N(x_0), \quad \text{where } D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-y)\, D_N(y)\, dy,$$

and $$f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0)\, D_N(y)\, dy$$

(using $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(y)\, dy = 1$)

Hence

$$f(x_0) - S_N f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f(x_0) - f(x_0 - y) \big)\, D_N(y)\, dy .$$

Now we define $g : [-\pi, \pi] \to \mathbb{R}$ by

$$g(y) = \begin{cases} \dfrac{f(x_0) - f(x_0 - y)}{y} & \text{if } \quad y \neq 0 \\ f'(x_0) & \quad y = 0 \end{cases}$$

Then g is unif bdd on $[-\pi, \pi]$. To see this, notice that $g(y) \to f'(x_0)$ as $y \to 0$ since f is diff at x_0

Hence $\exists\ \delta > 0$ such that

$$|g(y)| \leq |f'(x_0)| + 1 \qquad \text{as} \quad |y| < \delta$$

But for $y \in [-\pi, \pi] \setminus (-\delta, \delta)$,

$$|g(y)| = \left| \frac{f(x_0) - f(x_0 - y)}{y} \right|$$

$$\leq \frac{|f(x_0) - f(x_0 - y)|}{\delta}$$

$$\leq \frac{2 \cdot \|f\|_\infty}{\delta}.$$

Thus g is uniformly bounded on $[-\pi, \pi]$.

Also g is cts at almost all $y \in [-\pi, \pi]$. Hence $g \in \mathcal{R}$.

Notice that

$$\int_{-\pi}^{\pi} \left(f(x_0) - f(x_0 - y) \right) D_N(y)\, dy$$

$$= \int_{-\pi}^{\pi} g(y) \cdot y\, D_N(y)\, dy.$$

Notice that

$$y \cdot D_N(y) = y \cdot \frac{\sin (N + \frac{1}{2}) y}{\sin \frac{y}{2}}$$

$$= \frac{y}{\sin \frac{y}{2}} \cdot \left(\sin Ny \cdot \cos \frac{y}{2} + \cos Ny \sin \frac{y}{2} \right)$$

$$= \frac{y}{\sin\frac{y}{2}} \cdot \cos\frac{y}{2} \cdot \sin Ny$$

$$+ \frac{y}{\sin\frac{y}{2}} \cdot \sin\frac{y}{2} \cdot \cos Ny.$$

So

$$g(y)\cdot y\cdot D_N(y)$$

$$= g(y)\frac{y}{\sin\frac{y}{2}} \cdot \cos\frac{y}{2} \cdot \sin Ny$$

$$+ g(y)\cdot\frac{y}{\sin\frac{y}{2}} \sin\frac{y}{2} \cdot \cos Ny$$

$$= g_1(y)\sin Ny + g_2(y)\cos Ny$$

where $g_1(y) = g(y)\cdot \dfrac{y}{\sin\frac{y}{2}} \cos\dfrac{y}{2}$

$$g_2(y) = g(y)\cdot\frac{y}{\sin\frac{y}{2}} \sin\frac{y}{2}$$

Both g_1, g_2 are integrable.

By Riemann-Lebesgue Lemma,

$$\int_{-\pi}^{\pi} g_1(y)\cos Ny + g_2(y)\sin Ny \, dx \to 0$$

as $N \to +\infty$.

Hence $$\int_{-\pi}^{\pi} \left(f(x_0) - f(x_0 - y)\right) D_N(y)\, dy$$

$\to 0$ as $N \to \infty$.

That is,

$$f(x_0) - S_N f(x_0) \to 0 \quad \text{as } N \to \infty$$

$\square$